

Série 3a Solutions

Exercise 3a.1 – Tensile test on rods

In a standard tensile test, a steel rod of 50 mm in diameter and 2000 mm long is subjected to a tension force $F_1 = 47.1 \text{ kN}$. Knowing that $\nu = 0.3$ and $E_{st} = 200 \text{ GPa}$ determine:

- The elongation of the steel rod
- Change in diameter in the steel rod

Then a second bar is tested, it consists of a steel and an aluminium section, as shown below. When an axial force F is applied to the system, a strain gauge attached to the aluminium indicates an axial strain of 0.000873. The shear modulus of aluminium is $G_{alu} = 26.9 \text{ GPa}$. The aluminium's Poisson's ratio can be considered similar to steel with $\nu = 0.3$

- Determine the magnitude of the applied force F
- Find the total elongation of the steel and aluminium bar.

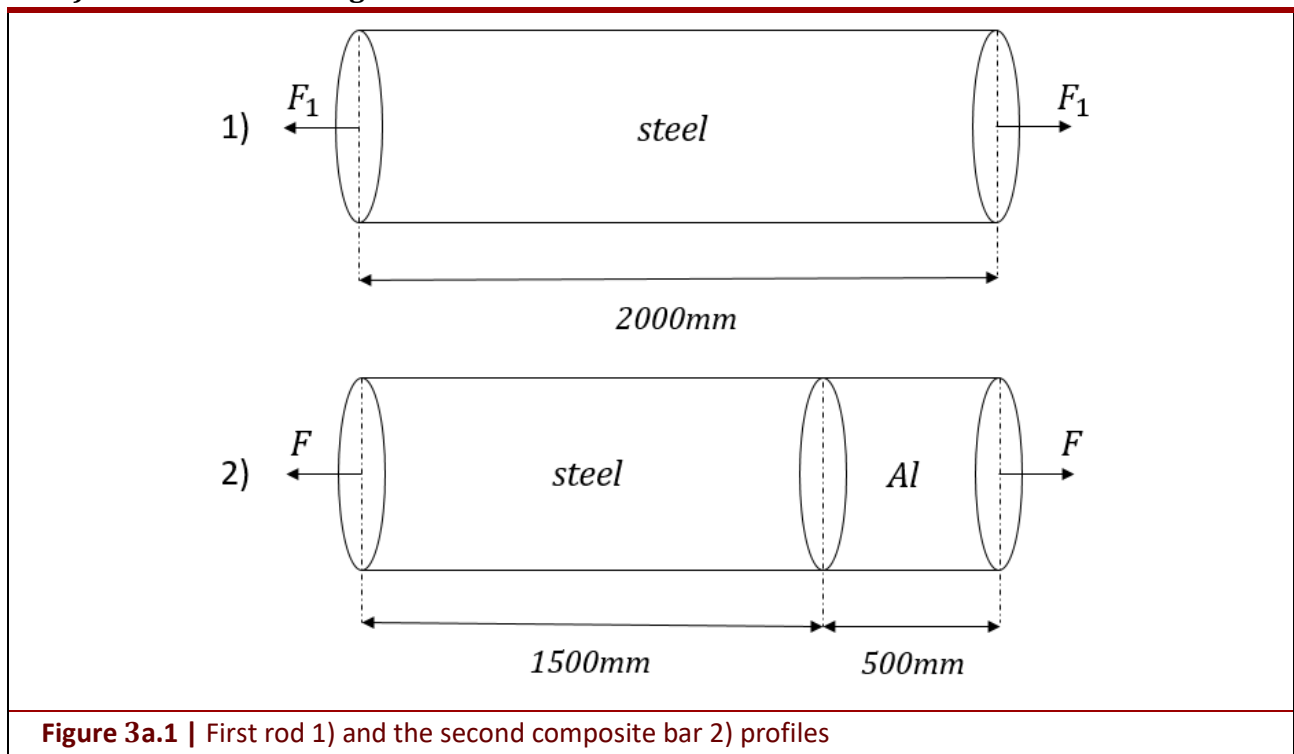


Figure 3a.1 | First rod 1) and the second composite bar 2) profiles

Solution 3a.1

Free body diagram

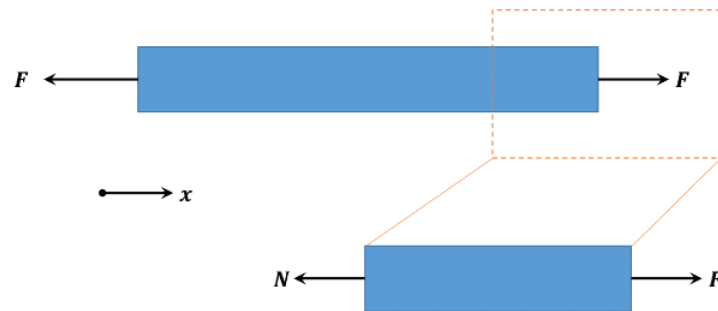


Figure 3a.1.1 | Free Body Diagram

Questions a) and b)

Objectives – what is asked?

- Total elongation of the steel rod
- Change of diameter of the steel rod

What is given?

- Geometry, length of material, diameter of the bar
- Poisson ratio
- Young's modulus for steel

Principles and formula

Displacement to elongation for 1D systems

$$\frac{\Delta L}{L} = \varepsilon_x \quad (1)$$

Hooke's law for 1D system

$$\sigma_x = E \cdot \varepsilon_x \quad (2)$$

Stress to load

$$\sigma_x = \frac{N}{A} \text{ where } A = \frac{\pi d^2}{4} \quad (3)$$

Equilibrium condition

$$F = N$$

Calculations

$$\sigma_x = \frac{N}{A} = \frac{4 * 47.1 * 10^3 N}{(0.05m)^2 * 3.14} = 24 \text{ MPa} \quad (4)$$

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{24 \text{ MPa}}{200 \text{ GPa}} = 1.2 * 10^{-4} \quad (5)$$

$$\Delta L = \varepsilon_x * L \Rightarrow 1.2 * 10^{-4} * 2 \text{ m} = 240 \text{ } \mu\text{m} \quad (6)$$

$$\varepsilon_r = -\nu \varepsilon_x = -0.3 * 1.2 * 10^{-4} = -3.6 * 10^{-5} \quad (7)$$

$$\Delta d = \varepsilon_r * d = -3.6 * 10^{-5} * 0.05 \text{ m} = -1.8 \text{ } \mu\text{m} \quad (8)$$

Questions c) and d)

Objectives – what is asked?

Value of applied load (F)

Total elongation

What is given?

Geometry, length of materials, diameter of the bar

Strain for aluminium

Young's moduli for steel and aluminium

Principles and formula

Displacement to elongation for 1D systems

$$\frac{\Delta L}{L} = \varepsilon_x$$

Hooke's law for 1D system

$$\sigma_x = E \cdot \varepsilon_x$$

General Hooke's law

$$2(1 + \nu) = \frac{E}{G} \quad (9)$$

Stress to load

$$\sigma_x = \frac{N}{A} \text{ with } F = N \text{ at equilibrium}$$

Calculations

From Hooke's law we have:

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{N}{AE} = \frac{N}{2AG(1 + \nu)} \quad (10)$$

Which implies that F can be deduced from the aluminium bar since the strain is given:

$$F = N = A \cdot G_{Al} \cdot 2(1 + \nu) \cdot \varepsilon_{Al} \quad (11)$$

We can then obtain the total elongation by adding the elongation of each segment:

$$\Delta L = \Delta L_{St} + \Delta L_{Al} = \varepsilon_{St} L_{St} + \varepsilon_{Al} L_{Al} \quad (12)$$

The strain in the aluminium segment is known (ε_{Al}) and the stress in steel can be calculated by noticing that the internal force N is constant along the bar and that and by combining Eq. (1) and Eq. (2) where steel and aluminium are considered to have similar Poisson's ratio:

$$\Delta L = \frac{\sigma_{St}}{E_{St}} L_{St} + \varepsilon_{Al} L_{Al} = \frac{N}{A \cdot E_{St}} L_{St} + \varepsilon_{Al} L_{Al} = \frac{G_{Al} \cdot 2(1 + \nu)}{E_{St}} \varepsilon_{Al} L_{St} + \varepsilon_{Al} L_{Al} \quad (13)$$

Numerical applications

$$F = 3.14 \cdot 0.025^2 \cdot 26.9 \cdot 10^9 \cdot 2(1 + 0.3) \cdot 0.000873 = 120 \text{ kN} \quad (14)$$

$$\Delta L = \frac{26.9 \cdot 2(1 + 0.3)}{200} \cdot 0.000873 \cdot 1.5 + 0.000873 \cdot 0.5 = 0.895 \text{ mm} \quad (15)$$

Exercise 3a.2 - Axially loaded bar + hydrostatic load

- a) For the axial loading shown in Figure 3a.2, determine the change in height and the change in volume of the brass cylinder shown.
- b) Solve the part a) assuming that the loading is hydrostatic with a $\sigma_x = \sigma_y = \sigma_z = -70 \text{ MPa}$.

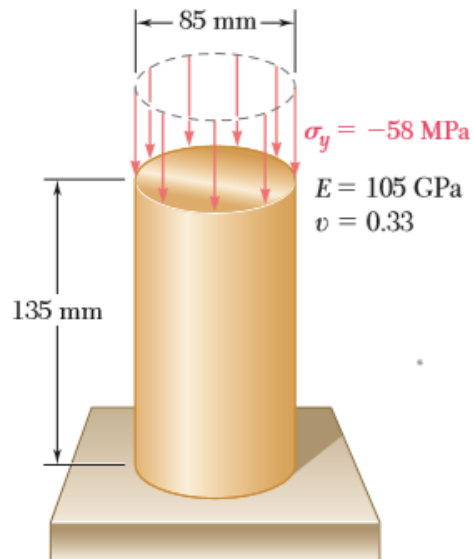


Figure 3a.2 | Axially loaded bar

Solution 3a.2

Let's calculate first the geometrical values:

$$h_0 = 135 \text{ mm} = 0.135 \text{ m} \quad (1)$$

$$A_0 = \frac{\pi}{4} d_0^2 = \frac{\pi}{4} * (85)^2 = 5.6745 * 10^3 \text{ mm}^2 = 5.674 * 10^{-3} \text{ m}^2 \quad (2)$$

$$V_0 = A_0 h_0 = 766.06 * 10^3 \text{ mm}^3 = 766.1 * 10^{-6} \text{ m}^3 \quad (3)$$

a) From the image we know that:

$$\sigma_x = 0, \sigma_y = -58 * 10^6 \text{ Pa and } \sigma_z = 0 \quad (4)$$

$$\epsilon_y = \frac{1}{E} (-\nu\sigma_x + \sigma_y - \nu\sigma_z) = \frac{\sigma_y}{E} = -\frac{58 * 10^6}{105 * 10^9} = -552.4 * 10^{-6} \quad (5)$$

The change in height is then calculated

$$\Delta h = h_0 \epsilon_y = 135 \text{ mm} * -552.4 * 10^{-6} = -0.075 \text{ mm} \quad (6)$$

Where the change of volume is:

$$\begin{aligned} \frac{\Delta V}{V} &= \epsilon_x + \epsilon_y + \epsilon_z = \frac{1 - 2\nu}{E} (\sigma_x + \sigma_y + \sigma_z) = \frac{1 - 2\nu}{E} (\sigma_y) \\ &= 0.34 * \frac{(-58 * 10^6)}{105 * 10^9} = -187.8 * 10^{-6} \end{aligned} \quad (7)$$

$$\Delta V = -187.8 * 10^{-6} * 766.1 * 10^{-6} \simeq -144 \text{ mm}^3 \quad (8)$$

b) Now the loads applied on the bar are: $\sigma_x = \sigma_y = \sigma_z = -70 * 10^6 \text{ Pa}$

$$\epsilon_y = \frac{1}{E} (-\nu\sigma_x + \sigma_y - \nu\sigma_z) = \frac{1 - 2\nu}{E} (\sigma_y) = 0.34 * \frac{(-70 * 10^6)}{105 * 10^9} = -226.7 * 10^{-6} \quad (9)$$

$$\Delta h = h_0 \epsilon_y = 135 \text{ mm} * -226.7 * 10^{-6} \simeq -0.031 \text{ mm} \quad (10)$$

$$\frac{\Delta V}{V} = \epsilon_x + \epsilon_y + \epsilon_z = \frac{1 - 2\nu}{E} (\sigma_x + \sigma_y + \sigma_z) = 3 * \frac{1 - 2\nu}{E} (\sigma_y) = -680 * 10^{-6} \quad (11)$$

$$\Delta V = -680 * 10^{-6} * 766.1 * 10^{-6} \simeq -521 \text{ mm}^3 \quad (12)$$

Exercice 3a.3 – Stress and strain on a constrained block

A block of magnesium alloy with $E = 45$ [GPa] and $\nu = 0.35$ is attached to a steel bracket, as shown in Figure 3a.3. Knowing that a load is applied which causes a stress of $\sigma_x = -180$ [MPa]. The length of the sides are given as $L_{x,0} = 0.1$ m, $L_{y,0} = 0.04$ [m] and $L_{z,0} = 0.025$ m.

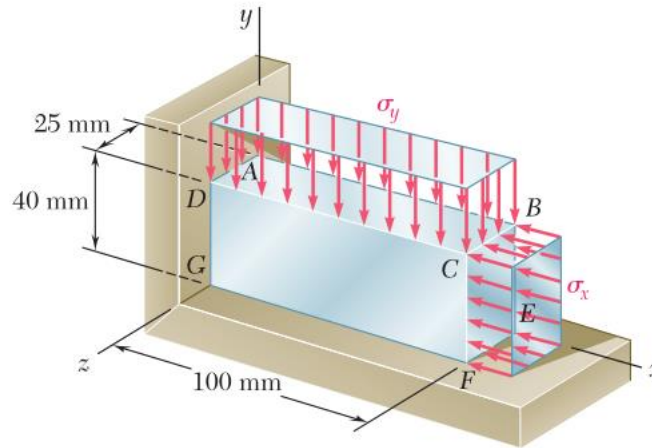


Figure 3a.3 | A magnesium alloy block attached to steel bracket

- Find the magnitude of σ_y for which the change in height of the block is zero.
- Find the corresponding change of the surface area of the face ABCD
- Find the corresponding change of volume for the block
- Suppose we want to keep the volume change as before but now the height can be changed. Instead of applying σ_x and σ_y , we apply two stresses $\sigma_x = -180$ [MPa] and σ_z . Calculate what the value of σ_z should be in the case where $\sigma_x = -180$ [MPa].

Solution 3a.3

a) Find the magnitude of σ_y for which the change in height of the block is zero.

We start from the y component of the compliance matrix as we are asked to find what normal stress on the y plane doesn't cause a strain in the y direction ($\varepsilon_y = 0$). Note that there are only two stresses applied on the body (σ_x and σ_y)

$$\varepsilon_y = 0 = \frac{1}{E}(-\nu\sigma_x + \sigma_y - \nu\sigma_z) \quad (1)$$

First we rewrite the formula in terms of σ_y

$$\sigma_y = \underbrace{E\varepsilon_y}_{=0} + \nu(\sigma_x + \underbrace{\sigma_z}_{=0}) \quad (2)$$

Then we can calculate the required stress in y to keep the strain zero

$$\sigma_y = \nu\sigma_x = 0.35 * (-180 * 10^6) = -63 * 10^6 = -63 \text{ [MPa]} \quad (3)$$

b) Find the corresponding change of the surface area of face ABCD

As the cube is only allowed to deform in the x and z directions we first need to determine the strains in these directions.

$$\varepsilon_x = \frac{1}{E}\{\sigma_x - \nu(\sigma_y + \sigma_z)\} = \frac{1}{E}\{-180 - 0.35(-63 + 0)\} * 10^6 = -0.0035 \quad (4)$$

$$\varepsilon_z = \frac{1}{E}\{\sigma_z - \nu(\sigma_x + \sigma_y)\} = \frac{1}{E}\{0 - 0.35(-180 - 63)\} * 10^6 = 0.0019 \quad (5)$$

We know that for a simple one dimensional case the change in length of a material is given as

$$\frac{\Delta L}{L_0} = \frac{L - L_0}{L_0} = \varepsilon \quad \text{or more simply} \quad L = L_0(1 + \varepsilon) \quad (6)$$

This also applies to higher dimensional cases such that we can determine the surface area of the deformed surface as

$$\Delta A + A_0 = L_x \cdot L_z = L_{x,0}(1 + \varepsilon_x) \cdot L_{z,0}(1 + \varepsilon_z) = L_{x,0}L_{z,0}(\varepsilon_x\varepsilon_z + \varepsilon_x + \varepsilon_z + 1) \quad (7)$$

Since A_0 is given from the start we can determine the change in surface area and filling in the values to obtain the change in area

$$\Delta A = L_x \cdot L_z - A_0 = L_{x,0}L_{z,0}(\varepsilon_x\varepsilon_z + \varepsilon_x + \varepsilon_z) \quad (8)$$

$$\Delta A = 0.1 * 0.025(-0.0035 * 0.0019 - 0.0035 + 0.0019) \approx -4.02 * 10^{-6} \text{ [m}^2\text{]} \quad (9)$$

c) Find the corresponding change of volume for the block

As we know that there is no change in the direction of y ($\varepsilon_y = 0$), we can simply multiply the change in the surface area by the initial height of the block $L_{y,0}$.

$$\Delta V = L_{y,0} \Delta A = 0.04 * -4.02 * 10^{-6} = -1.608 * 10^{-7} [\text{m}^3] \quad (10)$$

d) Suppose we want to keep the volume change as before but now the height can be changed. Instead of applying σ_x and σ_y , we apply two stresses $\sigma_x = -180$ [MPa] and σ_z . Calculate what the value of σ_z should be in the case where $\sigma_x = -180$ [MPa].

The volume change for the block is defined as

$$\Delta V = \{\varepsilon_x + \varepsilon_y + \varepsilon_z\}V \quad (11)$$

The formulas for the strain components are for this case, where $\sigma_y = 0$

$$\begin{aligned} \varepsilon_x &= \frac{1}{E} \{\sigma_x - \nu(\sigma_y + \sigma_z)\} = \frac{1}{E} \{\sigma_x - \nu\sigma_z\} \\ \varepsilon_y &= \frac{1}{E} \{\sigma_y - \nu(\sigma_x + \sigma_z)\} = -\frac{\nu}{E} \{\sigma_x + \sigma_z\} \\ \varepsilon_z &= \frac{1}{E} \{\sigma_z - \nu(\sigma_x + \sigma_y)\} = \frac{1}{E} \{\sigma_z - \nu\sigma_x\} \end{aligned} \quad (12)$$

This allows us to fill in the components in equation (11) to retrieve σ_z

$$\Delta V = \frac{1}{E} \{\sigma_x + \sigma_z - 2\nu\sigma_x - 2\nu\sigma_z\}V \quad (13)$$

$$\{\sigma_x + \sigma_z - 2\nu\sigma_x - 2\nu\sigma_z\} = \frac{\Delta VE}{V} \quad (14)$$

$$\sigma_x(1 - 2\nu) + \sigma_z(1 - 2\nu) = \frac{\Delta VE}{V} \quad (15)$$

$$\sigma_z(1 - 2\nu) = \frac{\Delta VE}{V} - \sigma_x(1 - 2\nu) \quad (16)$$

$$\sigma_z = \frac{\Delta VE}{V(1 - 2\nu)} - \sigma_x \quad (17)$$

If we then fill in the known values using the data for the volume from c), we find the value for σ_z :

note: $\Delta V = -1.62 * 10^{-7} \text{ m}^3$ when using the values σ to compute it.

$$\begin{aligned} \sigma_z &= \frac{-1.608 * 10^{-7} * 45 * 10^9}{(0.1 * 0.04 * 0.025)(1 - 2 * 0.35)} - (-180 * 10^6) \\ \sigma_z &= -241.2 * 10^6 + 180 * 10^6 = -61.2 \text{ [MPa]} \end{aligned} \quad (18)$$

Exercise 3a.4 – Multi-segment bar

Consider the following bar structure (Figure 3a.4) with force $3F$ acting upon point B and force F acting upon point D. Section CD is made of two identical bars in parallel of area A and Young's modulus E that are secured by rigid plates. Section AC consists of a single bar of area A and Young's modulus E . **What is the overall displacement of Point D, and the internal stress in one of the bars in section CD?**

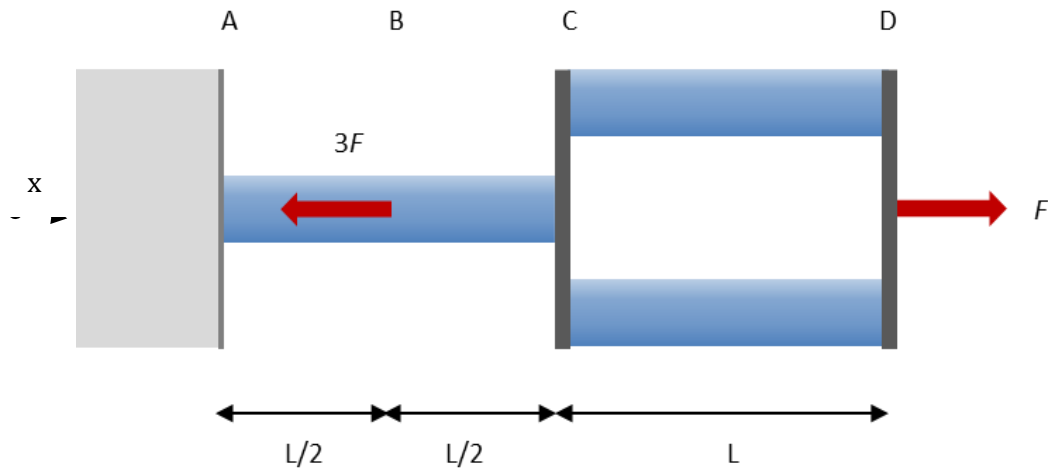


Figure 3a.4 | Multi-segment structure.

Solution 3a.4

What is given?

Bars cross-section areas: A

Applied forces: F and $3F$

Young's modulus E

What is asked?

Displacement of point D Δ_{Total} and internal stress in region CD.

Region CD

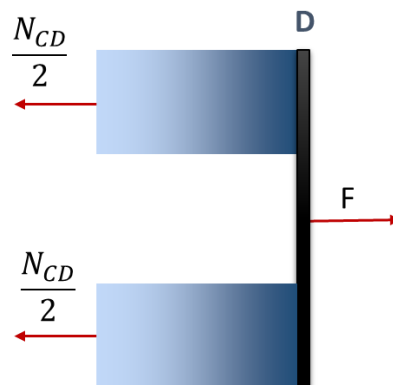


Figure 3a.4.1 | Sliced section in the multi-segment bar

What are the Eqs. that are required?

The stiffness of a segment AB:

$$k_{AB} = \frac{AE}{L_{AB}} \quad (1)$$

Where A is the cross-section area of segment AB, L_{AB} , is the length and E the Young's modulus.

The internal force of a segment AB with respect to the displacement:

$$N_{AB} = k_{AB} \Delta_{AB} \quad (2)$$

The internal stress of a substructure:

$$\sigma = \frac{N}{A} \quad (3)$$

Find Reaction at Point A

$$\sum F_x = 0 \quad (4)$$

$$R_A - 3F + F = 0 \rightarrow R_A = 2F \quad (5)$$

Applying the *Method of Sections* to Region AB we can easily see that there is an internal force of $N_{AB} = -2F$.

Region AB

Stiffness of Region AB

$$k_{AB} = \frac{AE}{\frac{L}{2}} = \frac{2AE}{L} \quad (6)$$

Calculate elongation of Region AB

$$N_{AB} = k_{AB}\Delta_{AB} \quad (7)$$

$$\Delta_{AB} = \frac{-2F}{\left(\frac{2AE}{L}\right)} = \frac{-FL}{AE} \quad (8)$$

Region BD

Stiffness of Region BD

$$\frac{1}{k_{BD}} = \frac{1}{k_{BC}} + \frac{1}{k_{CD} + k_{CD}} \quad (9)$$

$$\frac{1}{k_{BD}} = \frac{1}{\frac{AE}{\left(\frac{L}{2}\right)}} + \frac{1}{\frac{AE}{L} + \frac{AE}{L}} \rightarrow k_{BD} = \frac{AE}{L} \quad (10)$$

Calculate Displacement of Region BD

Applying the *Method of Sections* to Region BD we can easily find that $N_{BD} = F$

$$N_{BD} = k_{BD}\Delta_{BD} \quad (11)$$

$$\Delta_{BD} = \frac{F}{\left(\frac{AE}{L}\right)} = \frac{FL}{AE} \quad (12)$$

Superimpose all Displacements

$$\Delta_{Total} = \Delta_{AB} + \Delta_{BD} \quad (13)$$

$$\Delta_{Total} = \frac{-FL}{AE} + \frac{FL}{AE} = 0 \quad (14)$$

Calculate the stress in region CD:

$$\sigma_{CD} = \frac{N_{CD}/2}{A} = \frac{F/2}{A} = \frac{F}{2A} \quad (15)$$

Exercise 3a.5 – Elongation of a composite bar

We consider the circular bar of Figure 3a.5, composed of two materials. The material properties are: $E_A = \frac{10}{9} GPa$, $E_B = 2.5 GPa$, $\nu_A = \nu_B = 0.25$; and the dimensions: $L = 1 m$, $R = 5 mm$, $r = 3 mm$. We apply a force $F = 360\pi N$ at the free-end of the bar:

- Calculate the total elongation of the bar.
- Calculate the strain in materials A and B when $x = \frac{L}{2}$. Give a numerical value.
- Calculate the stress in materials A and B when $x = \frac{L}{2}$. Give a numerical value.

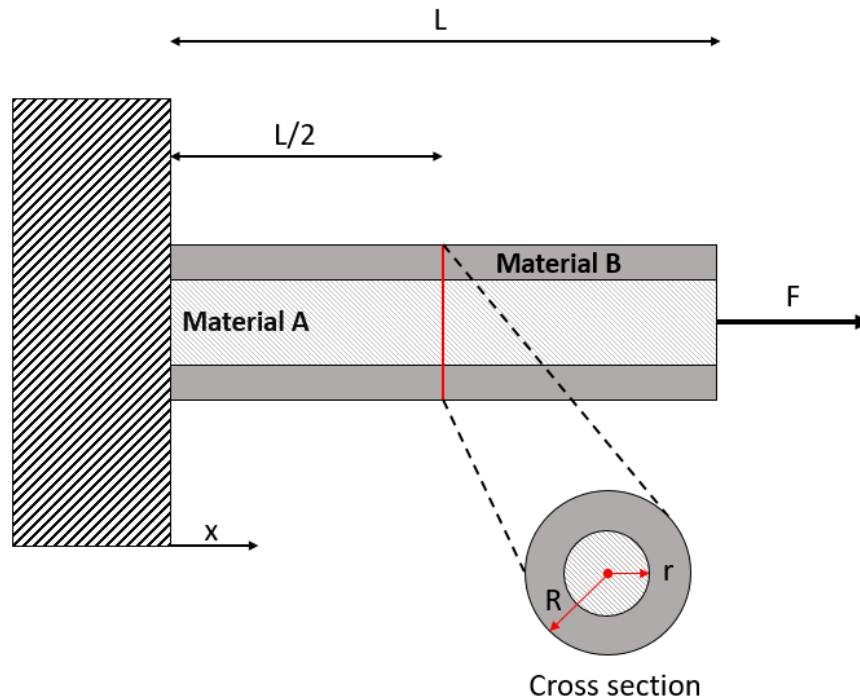
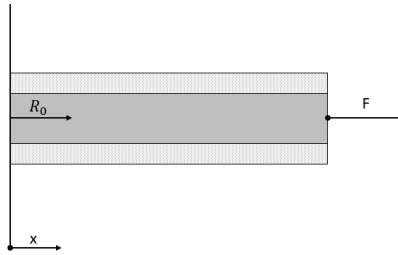


Figure 3a.5 | The composite bar with a vertical section of its profile

Solution 3a.5

Apply force equilibrium Eq. to the entire structure



$$\sum F_x = 0 \rightarrow R_0 + F = 0 \rightarrow R_0 = -F \quad (1)$$

(a) Calculate the total elongation of the bar

For the evaluation of the total elongation of the bar both flexibility and stiffness method can be used. Reminding that:

$$k = \frac{1}{f} = \frac{EA}{L} \quad (2)$$

$$F = k_{eq} \delta \rightarrow \delta = \frac{F}{k_{eq}} \quad (3)$$

Since it's a composite bar, made by two different bars in parallel, we can write the equivalent stiffness as:

$$k_{eq} = k_A + k_B = \frac{E_A A_A}{L} + \frac{E_B A_B}{L} \quad (4)$$

Where A_A and A_B are the areas of the sections relative to the two materials.

So we can write the full equation to obtain the elongation:

$$\delta = \frac{F}{k_{eq}} = \frac{F}{\frac{E_A A_A}{L} + \frac{E_B A_B}{L}} = \frac{FL}{E_A A_A + E_B A_B} \quad (5)$$

$$= \frac{360\pi N \cdot 1m}{(\pi \cdot 9 \cdot 10^{-6} m^2 \cdot \frac{10}{9} \cdot 10^9 Pa) + (\pi \cdot 16 \cdot 10^{-6} m^2 \cdot 2.5 \cdot 10^9 Pa)} = 7.2 \cdot 10^{-3} m$$

(b) Calculate the strain in materials A and B when $x = \frac{L}{2}$

Since the composition of the composite bar do not change along the x axis, the strain is not dependent from x, $\varepsilon(x) = \text{constant}$.

Reminding that:

$$\varepsilon = \frac{\delta}{L_0} \quad (6)$$

And that because the bar is a composite bar, the cylinder and the tube must shorten by the same amount and by indicating the two shortenings with δ_B and δ_A , we obtain the following equation of compatibility:

$$\delta_A = \delta_B \quad (7)$$

We can deduce that

$$\varepsilon_A = \varepsilon_B \quad (8)$$

Since the total elongation of the bar has been calculated in (a) and $L_0 = 1m$

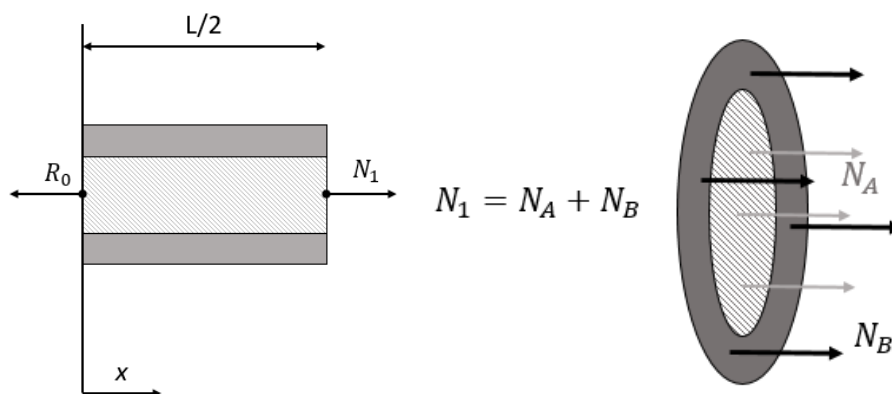
$$\delta = \delta_A = \delta_B$$

We can write:

$$\varepsilon_A = \varepsilon_B = \frac{\delta}{L_0} = \frac{7.2 \cdot 10^{-3}m}{1m} = 0.0072 \quad (9)$$

(c) Calculate the stress in materials A and B when $x = \frac{L}{2}$

For section at $x=L/2$



The force N_1 that is uniformly distributed along all the cross section of the bar, it can be decomposed in:

$$N_1 = N_A + N_B = 360\pi N \quad (10)$$

Where the force N_A is the resultant of the uniformly distributed stresses acting over the cross section of the inner cylinder and the force N_B is the resultant of the stresses acting over the cross section of the tube.

As already said, the cylinder and the tube must shorten by the same amount.

$$\delta_A = \delta_B$$

The general formula for δ is:

$$\delta = \frac{FL}{EA} \quad (11)$$

By substituting we can obtain the values of the force applied on each part of the bar: N_B and N_A .

$$\begin{aligned} \delta_A &= \frac{N_A L}{E_A A_A} \\ \delta_B &= \frac{N_B L}{E_B A_B} \\ \frac{N_A}{E_A A_A} &= \frac{N_B}{E_B A_B} \end{aligned} \quad (12)$$

Combined with:

$$N_1 = N_A + N_B$$

Give us :

$$\begin{aligned} N_A &= N_1 \left(\frac{E_A A_A}{E_A A_A + E_B A_B} \right) = N_1 \left(\frac{A_A}{A_A + \frac{E_B}{E_A} A_B} \right) \\ &= 360\pi \left(\frac{9\pi}{9\pi + \frac{2.5}{10} 16\pi} \right) = 360\pi \cdot \frac{1}{5} = 72\pi = 226N \end{aligned} \quad (13)$$

$$\begin{aligned} N_B &= N_1 \left(\frac{E_B A_B}{E_A A_A + E_B A_B} \right) = N_1 \left(\frac{A_B}{A_B + \frac{E_A}{E_B} A_A} \right) \\ &= 360\pi \left(\frac{16\pi}{16\pi + \frac{10}{9} 9\pi} \right) = 360\pi \cdot \frac{4}{5} = 288\pi = 904N \end{aligned} \quad (14)$$

i. **In the inner core (material A)**

$$\sigma_A = \frac{N_A}{A_A} = \left(\frac{N_1}{A_A + \frac{E_B}{E_A} A_B} \right) = \frac{72\pi}{\pi \cdot 9 \cdot 10^{-6}} = 8MPa \quad (15)$$

$$\sigma_A = \varepsilon \cdot E_A = 8 \cdot 9 \cdot 10^{-4} \cdot \frac{10}{9} \cdot 10^9 = 8MPa \quad (16)$$

ii. **In the shell (material B)**

$$\sigma_B = \frac{N_B}{A_B} = \left(\frac{N_1}{A_B + \frac{E_A}{E_B} A_A} \right) = \frac{288\pi}{\pi \cdot 16 \cdot 10^{-6}} = 18MPa \quad (17)$$

$$\sigma_B = \varepsilon \cdot E_B = 8 \cdot 9 \cdot 10^{-4} \cdot \frac{5}{2} \cdot 10^9 = 18MPa \quad (18)$$

Exercise 3a.6 – Plane Strain

In many situations physical constraints prevent strain from occurring in a given direction. For example, $\varepsilon_z = 0$ in the case shown in Figure 3a.6, where longitudinal movement of the long prism is prevented at every point. Plane sections perpendicular to the longitudinal axis remain plane and at the same distance apart.

Show that for this situation (i.e. $\varepsilon_z = 0$) which is known as plane strain the following equations are true.

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\varepsilon_x = \frac{1}{E} [(1 - \nu^2)\sigma_x - \nu(1 + \nu)\sigma_y]$$

$$\varepsilon_y = \frac{1}{E} [(1 - \nu^2)\sigma_y - \nu(1 + \nu)\sigma_x]$$

Hint: Start with the generalized Hooke's Law for strain.

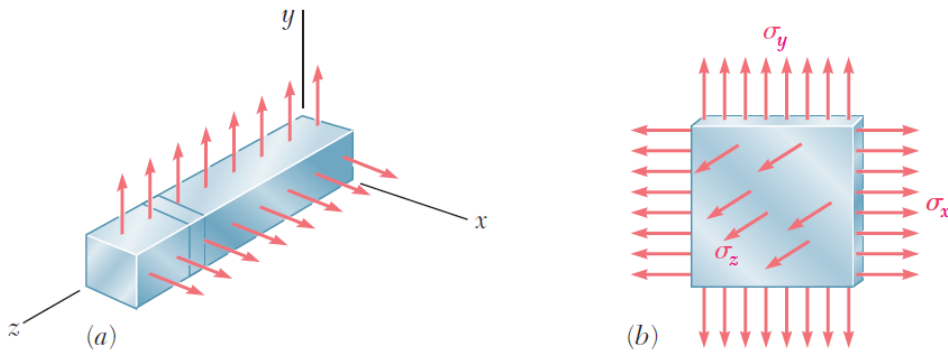


Figure 3a.6 | (a) Plane strain on a prism, (b) cross-section of the prism

Solution 3a.6

We are given prism with σ_x and σ_y that yields $\varepsilon_z = 0$. From the generalized Hooke's Law for strain, we can derive the equations for σ_z , ε_x , and ε_y . First start with compliance matrix and then substitute known values from the system in order to achieve the above proof.

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \quad (1)$$

From the third row of matrix we have the following

$$\begin{aligned} \varepsilon_z = 0 &= \frac{1}{E} (-\nu\sigma_x - \nu\sigma_y + \sigma_z) \\ \sigma_z &= \nu(\sigma_x + \sigma_y) \end{aligned} \quad (2)$$

From the first row of matrix we have the following

$$\varepsilon_x = \frac{1}{E} (\sigma_x - \nu\sigma_y - \nu\sigma_z) \quad (3)$$

Including the expression provided for σ_z we can find the following

$$\varepsilon_x = \frac{1}{E} (\sigma_x - \nu\sigma_y - \nu^2(\sigma_x + \sigma_y)) = \frac{1}{E} ((1 - \nu^2)\sigma_x - \nu(1 + \nu)\sigma_y) \quad (4)$$

From the second row of matrix we have the following

$$\varepsilon_y = \frac{1}{E} (-\nu\sigma_x + \sigma_y - \nu\sigma_z) \quad (2)$$

Including the expression provided for σ_z we can find the following

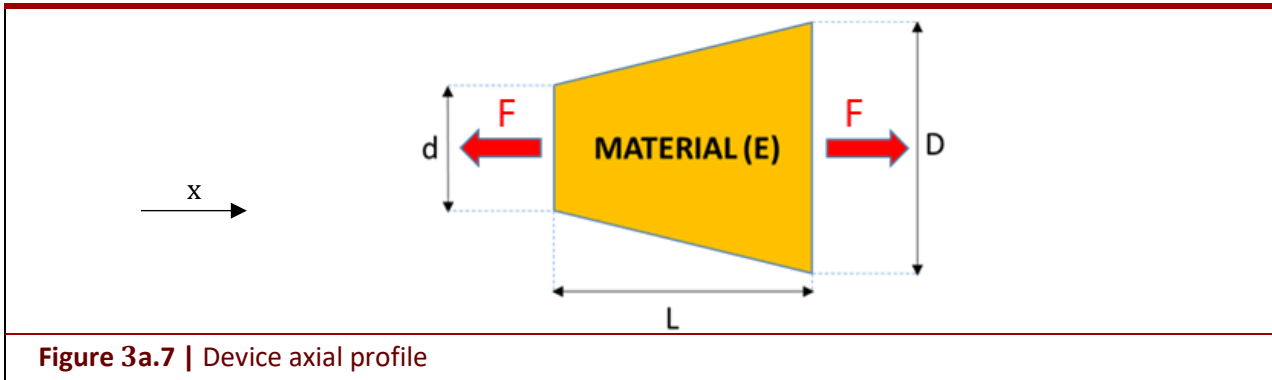
$$\varepsilon_y = \frac{1}{E} (-\nu\sigma_x + \sigma_y - \nu^2(\sigma_x + \sigma_y)) = \frac{1}{E} ((1 - \nu^2)\sigma_y - \nu(1 + \nu)\sigma_x) \quad (6)$$

OPTIONAL - Exercise 3a.7- Trapezoidal bar

Consider a trapezoidal bar with a circular cross-section. Its dimensions and Young modulus are known (As shown in Figure 3a.7). We apply an axial load F at its ends.

Calculate the elongation δ of this structure.

$$\text{Mathematical Hint : } \int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}$$



Solution 3a.7

What is given?

Length: L

Short-end diameter: d

Large-end diameter: D

Young Modulus: E

Applied Load: F

Assumptions

The material is isotropic and we work at constant temperature

What is asked?

Elongation δ

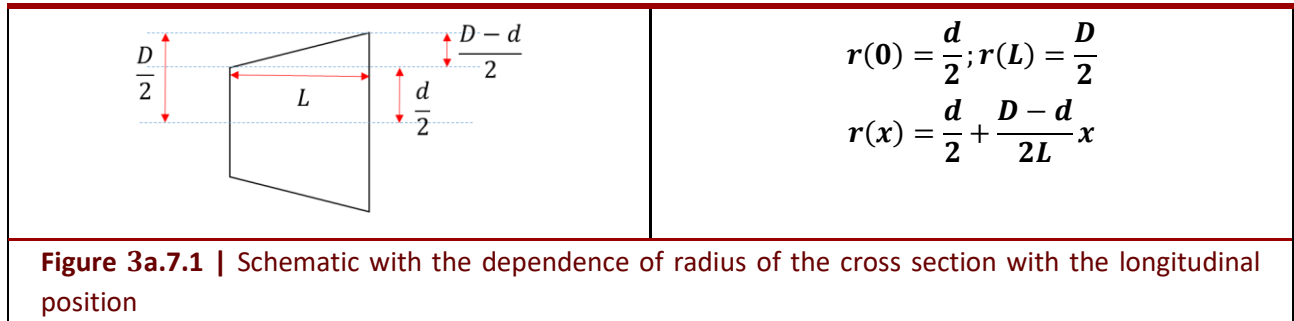
Principles and formula

Elongation can be calculated from the microscopic definition of strain:

$$\varepsilon_x = \frac{\partial u}{\partial x} \rightarrow du = \varepsilon_x(x) \cdot dx \rightarrow \delta = \int_0^L \varepsilon_x(x) dx = \int_0^L \frac{N(x)}{E(x)A(x)} dx \quad (1)$$

Where $N(x)$, $E(x)$ and $A(x)$ are respectively the internal force, Young's modulus and cross-sectional area of the part of the bar at a distance x from the origin; and δ is the overall elongation. $A(x)$ is the cross-sectional area along the longitudinal axis, which is given by:

$$A(x) = \pi \cdot r(x)^2 = \pi \left(\frac{d}{2} + \frac{D-d}{2L}x \right)^2 \quad (2)$$



Perform calculations in symbolic form

Since the applied load and the Young modulus are constant along the material we can extract them from the integral of Eq. (1) using Eq. (2):

$$\begin{aligned} \delta &= \frac{F}{E} \int_0^L \frac{dx}{A(x)} = \frac{F}{\pi E} \int_0^L \frac{dx}{\left(\frac{d}{2} + \frac{D-d}{2L}x \right)^2} = -\frac{F}{\pi E} \frac{2L}{D-d} \left(\frac{1}{\left(\frac{d}{2} + \frac{D-d}{2L}L \right)} - \frac{1}{\left(\frac{d}{2} \right)} \right) = \\ &= -\frac{F}{\pi E} \frac{4L}{D-d} \left(\frac{d-D}{dD} \right) = 4 \frac{F}{\pi E} \frac{L}{dD} \end{aligned} \quad (3)$$

Check your physical dimensions

$$[\delta] = [Length] = \frac{[Force]}{[Pressure]} * \frac{[Length]}{[Length]^2} = [Length] \quad (4)$$